CONNECTIONS ON METRIPLECTIC MANIFOLDS

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Abstract. In this note we discuss conditions under which a linear connection on a manifold equipped with both a symmetric (Riemannian) and a skew-symmetric (almost-symplectic or Poisson) tensor field will preserve both structures.

If (M, g) is a (pseudo-)Riemannian manifold, then classical results due to T. Levi-Civita, H. Weyl and E. Cartan [7] show that for any $(1, 2)$ tensor field T^i_{jk} which is skew-symmetric by lower indices, there exists a unique linear connection Γ preserving the metric ($\nabla^\Gamma g = 0$), with T as its torsion tensor: $T^i_{kj} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj})$.

It has also been shown [4] that given any symmetric (by lower indices) $(1, 2)$ tensor S^i_{jk} on a symplectic manifold (M, ω), there exists a unique linear connection preserving ω which has S as its symmetric part, i.e., $S^i_{jk} = \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj})$. Moreover, it is known [9] that if ω is a regular Poisson tensor on M, then there always exists a linear connection on M with respect to which ω is covariantly constant. Such connections are called *Poisson* connections, and can be chosen to coincide with the Levi-Civita connection of the metric g (if g is Riemannian) in certain cases.

Considering these results, one is naturally led to the question: Given a skew-symmetric $(0, 2)$ tensor ω, and a (pseudo-)Riemannian metric g on a manifold M, when do there exist linear connections preserving ω and g simultaneously:

$$\nabla^\Gamma \omega + \nabla^\Gamma g = 0 \ ? \tag{1}$$

Motivated by the terminology of P.J. Morrison [6], we call the a manifold equipped with both a (pseudo-)Riemannian metric g and a skew-symmetric $(2, 0)$ tensor P a *metriplectic* manifold, and a connection which preserves both tensors will be called a *metriplectic connection*. In the first section we restrict ourselves to the case in which both $\omega = P^{-1}$ and g are nondegenerate, that is ω is almost-symplectic and g is Riemannian. We combine the results from [7] and [4] to derive a necessary condition for a connection Γ to be a metriplectic connection. We also discuss the form of Γ in the almost-Hermitian and symplectic cases. The main result of this section is the following

Proposition *Any connection Γ with symmetric part Π and torsion T that preserves both a Riemannian metric g and an (almost-)symplectic form ω has the form $\Gamma = \Pi + T$*

$$\Pi = L(g) + \bar{g}(T), \quad T = \omega^{-1}\nabla^g \omega - (1/2)\omega^{-1} d\omega - \bar{\omega}\bar{g}(T),$$

where $L(g)$ is the Levi-Civita connection defined by the metric g, and the "bar" operators $\bar{g}(T)$ and $\bar{\omega}\bar{g}(T)$ are related to the symmetries of the torsion T.

In the second section we give the proof of a theorem due to Shubin [8] which states that if M admits a metriplectic connection, and $P = \omega^{-1}$ is nondegenerate, then M is a Kähler manifold. We also formulate an observation made by Vaisman [9] as a generalization of Shubin's theorem in the case that P is degenerate.

1. Necessary Conditions on the Metriplectic Connection Γ

We consider here the case where $\omega = P^{-1}$ is skew-symmetric and nondegenerate (not necessarily closed), and g is Riemannian. Suppose that $\Gamma = \Pi + T$ is a connection on the Poisson-Riemannian manifold (M, g, ω) with Π and T symmetric and skew-symmetric (torsion) tensors respectively. Assume that Γ satisfies (1). Since $\nabla^\Gamma g = 0$, we know from [4] that the symmetric tensor Π must have local components

$$\Pi^i_{jk} = L(g)^i_{jk} + g^{is}(T^q_{sj} g_{qk} + T^q_{sk} g_{jq}), \tag{2}$$

where $L(g)^i_{jk} = (1/2)g^{is}(g_{js,k} + g_{ks,j} - g_{jk,s})$ is the Levi-Civita connection for g. On the other hand, since $\nabla^\Gamma \omega = 0$, we have ([7])

$$T^i_{jk} = L(\omega)^i_{jk} - \omega^{is}(\Pi^q_{sj}\omega_{qk} + \Pi^q_{sk}\omega_{jq}), \tag{3}$$

where $L(\omega)^i_{jk} = (1/2)\omega^{is}(\omega_{js,k} + \omega_{sk,j} + \omega_{jk,s})$. We introduce the following operator: for any nondegenerate $(0, 2)$ tensor h define the linear operator $\bar{h}$ on $(1, 2)$ tensors by

$$\bar{h}(B)^i_{jk} = h^{is}(B^q_{sj} h_{qk} + B^q_{sk} h_{kq}).$$

With this definition, we can write (2) and (3) as

$$\Pi = L(g) + \bar{g}(T) \quad \text{and} \quad T = L(\omega) - \bar{\omega}(\Pi).$$

Thus, the original connection Γ has the form

$$\begin{aligned} \Gamma &= L(g) + L(\omega) - \bar{\omega}(\Pi) + \bar{g}(T), \\ &= L(g) + L(\omega) - \bar{\omega}(L(g)) - \bar{\omega}(\bar{g}(T)) + \bar{g}(T). \end{aligned} \tag{4}$$

Notice that when $\bar{h}$ operates on a connection form A, the result is related to the covariant derivative of h with respect to A as follows:

$$\nabla^A h = \partial h - h(\bar{h}(A)) \ \text{ or } \ \bar{h}(A) = h^{-1}\partial h - h^{-1}\nabla^A h. \tag{5}$$

In particular, $\bar{\omega}(L(g)) = \omega^{-1}(\partial\omega - \nabla^g \omega)$ where ∇^g is the covariant derivative with respect to g. So we have

$$\Gamma = L(g) + L(\omega) - \omega^{-1}(\partial\omega - \nabla^g\omega) - \bar{\omega}(\bar{g}(T)) + \bar{g}(T).$$

Rewriting $L(\omega)$ as $\omega^{-1}\partial\omega - (1/2)\omega^{-1}d\omega$, we have the following

Proposition. *Any connection Γ the preserves both a Riemnannian metric g and an (almost-)symplectic form ω has the form $\Gamma = \Pi + T$ with*

$$\Pi = L(g) + \bar{g}(T), \quad T = \omega^{-1}\nabla^g\omega - (1/2)\omega^{-1}d\omega - \bar{\omega}\bar{g}(T).$$

If ω is closed (i.e. (M, ω) is symplectic),then $\Gamma = L(g) + \bar{g}(T) + \omega^{-1}\nabla^g\omega - \bar{\omega}\bar{g}(T)$.

1.1. Almost-Hermitian Connections with Totally Skew Torsion.

Suppose now that g and ω are related by an almost-complex structure $J = g^{-1}\omega$ satisfying $J^2 = -I$. Observe that $\bar{g}(T) = 0$ if and only if T is *totally skew-symmetric* with respect to the metric g (that is, $T_{ijk} = T^q_{ij} g_{qk}$ is an exterior 3-form). In this case,

$$T = \omega^{-1}\nabla^g\omega - (1/2)\omega^{-1}d\omega. \tag{6}$$

Thus Γ reduces to the canonical almost-hermitian connection with totally skew torsion (see e.g. [3] [5]). Indeed, the torsion 3-form $T(X, Y, Z) =< T(X, Y), Z >_g$ can be expressed in terms of the Nijenhuis tensor $N(X, Y, Z) =< N(X, Y), Z >_g$ of the almost-complex structure J (see [2]) as follows.

PROPOSITION. *If the torsion of an almost-Hermitian connection* Γ *is totally skew-symmetric, then*

$$T(X,Y,Z) = (1/2)N(X,Y,Z) - (1/2)d\omega(JX,JY,JZ)$$

for all X,Y,Z.

Proof.

$$\begin{aligned}2T_{ijk} &= (2\omega^{qn}\nabla^g_n\omega_{ij} - \omega^{qn}d\omega_{ijn})g_{qk}\\ &= 2J^n_k\nabla^g_n\omega_{ij} - J^n_k(\nabla^g_n\omega_{ij} + \nabla^g_j\omega_{ni} + \nabla^g_i\omega_{jn})\\ &= J^n_k\nabla^g_n\omega_{ij} - J^n_k\nabla^g_j\omega_{ni} - J^n_k\nabla^g_i\omega_{jn}\\ &= J^n_k\nabla^g_n\omega_{ij} - J^n_i\nabla^g_j\omega_{kn} + J^n_j\nabla^g_i\omega_{kn}\end{aligned}$$

$$= -J^n_k\nabla^g_n\omega_{ji} - J^n_i\nabla^g_n\omega_{kj} - J^n_j\nabla^g_n\omega_{ik} + J^n_i\nabla^g_n\omega_{kj} - J^n_j\nabla^g_n\omega_{ki} - J^n_i\nabla^g_j\omega_{kn} + J^n_j\nabla^g_i\omega_{kn}$$

$$= -J^n_k\nabla^g_n\omega_{ji} - J^n_i\nabla^g_n\omega_{kj} - J^n_j\nabla^g_n\omega_{ik} + N_{ijk}.$$

Using the fact that $\omega_{in}\nabla^g_q J^n_j = -J^n_j\nabla^g_q\omega_{in}$, we have

$$J^n_k\nabla^g_n\omega_{ji} = J^n_k g_{jr}\nabla^g_n J^r_i = J^n_k(J^s_j\omega_{sr})\nabla^g_n J^r_i = -J^n_k J^s_j J^r_i\nabla^g_n\omega_{sr}.$$

Permuting the indices i,j,k gives us

$$\begin{aligned}2T_{ijk} &= (\nabla^g_t\omega_{sr} + \nabla^g_r\omega_{ts} + \nabla^g_s\omega_{rt})J^r_i J^s_j J^t_k + N_{ijk}\\ &= -d\omega_{rst}J^r_i J^s_j J^t_k + N_{ijk}.\end{aligned}$$

□

Using this expression for T together with (6), it is easy to see that M is:

Hermitian (J is integrable) $\leftrightarrow T(X,Y,Z) = -(1/2)d\omega(JX,JY,JZ)$,
Symplectic (ω is closed) $\leftrightarrow T = (1/2)N$, and
Kähler (ω is closed and g-parallel) $\leftrightarrow \Gamma$ is the Levi-Civita connection determined by g.

1.2. THE SYMPLECTIC CASE. As mentioned above, if M,ω is symplectic, then the connection Γ takes the form

$$\Gamma = L(g) + \bar{g}(T) + \omega^{-1}\nabla^g\omega - \bar{\omega}\bar{g}(T). \tag{7}$$

Applying (5) to ω and Γ, we see that the condition $\nabla^\Gamma\omega = 0$ is equivalent to

$$\begin{aligned}0 &= \partial\omega - \omega(\bar{\omega}(\Gamma))\\ &= \partial\omega - \omega(\bar{\omega}(\Pi)) - \omega(\bar{\omega}(T))\\ &= \partial\omega - \omega(\bar{\omega}(L(g))) - \omega(\bar{\omega}(\bar{g}(T))) - \omega(\bar{\omega}(T))\\ &= \nabla^g\omega - \omega(\bar{\omega}(\bar{g}(T))) - \omega(\bar{\omega}(T)).\end{aligned}$$

However, $0 = \nabla^g\omega - \omega(\bar{\omega}(\bar{g}(T))) - \omega(T)$. Thus we obtain the following condition on T:

$$\bar{\omega}(T) = T \quad \text{or} \quad (\bar{\omega} - I)T = 0.$$

REMARK. *This result can be derived directly from the condition* $\nabla^\Gamma\omega = 0$ *together with the Jacobi condition for* ω*, and is easily seen to be equivalent to the following cyclic condition on the indices of the tensor* $T\omega$*,*

$$T^s_{ij}\omega_{sk} + T^s_{ki}\omega_{sj} + T^s_{jk}\omega_{si} = 0.$$

Thus, we may substitute $\bar{g}(\bar{\omega}(T))$ for $\bar{g}(T)$ in (7). Writing the difference $\bar{g}(\bar{\omega}(T)) - \bar{\omega}(\bar{g}(T))$ as $[\bar{g},\bar{\omega}](T)$, we arrive at the following

Proposition. *A linear connection* Γ *on a symplectic-Riemannian manifold* (M, g, ω) *which preserves both* g *and* ω *has the form*

$$\Gamma = L(g) + \omega^{-1}\nabla^g\omega + [\bar{g}, \bar{\omega}](T).$$

2. Necessary Conditions on the structure of (M, g, P)

It is known [5] that if M is a Kähler manifold, then the Kähler form is parallel with respect to the Levi-Civita connection L on M defined by the Kähler metric, in which case (1) clearly holds (with $\Gamma = L$). In this section we will discuss a partial converse to this fact due to M. Shubin.

2.1. Shubin's Theorem. On a manifold M, let g_0 be a Riemannian metric and let $\omega = P^{-1}$ be an almost-symplectic form (non-degenerate and skew-symmetric). Let $L(g_0)$ denote the Levi-Civita connection associated with g_0, and let $K = g_0 + \omega$. We denote the covariant derivatives with respect and $L(g_0)$ by ∇^0. The following theorem is a reformulation of a result by Shubin [8], and its proof follows Shubin's proof, with some variations.

Theorem. *If* (M, g_0, ω) *is an almost-symplectic Riemannian manifold, and* $\nabla^0 K = 0$*, then there exists a complex structure* J *on* M *such that the metric* g *defined by* $g(X, Y) = \omega(X, JY)$ *is parallel with respect to* g_0 *(thus* $L(g) = L(g_0)$*), and defines a Kähler structure on* M*.*

Proof. First note that if $\nabla^0 K = 0$, then $\nabla^0\omega = 0$. Since ∇^0 is symmetric, it follows (see remark 1.4 in [4]) that $d\omega = 0$, and so ω is symplectic. Now, any symplectic manifold [8] admits an almost-complex structure J defined by $J = A(-A^2)^{-1/2}$, where A is the linear operator defined by

$$g_0(AX, Y) = \omega(X, Y).$$

Since both g_0 and ω are parallel with respect to ∇^0, it is clear that the operator A will also be parallel, thus $\nabla^0 J = 0$. The integrability of J then follows from the expression

$$N_J(X, Y) = (\nabla^0_{JX}J)Y - (\nabla^0_{JY}J)X + J(\nabla^0_Y J)X - J(\nabla^0_X J)Y$$

for the Nijenhuis torsion of J (see [1]).

The metric $g(X, Y) = \omega(X, JY)$ is Hermitian with respect to J. Therefore, it defines a Kähler structure on M. Furthermore, the equality $g_{ij} = \omega_{ik}J^k_j$ shows that $\nabla^0 g = 0$. The connection $L(g_0)$ is symmetric and compatible with g, so it must coincide with the Levi-Civita connection $L(g)$. □

2.2. Generalization to a degenerate P. If the tensor P is degenerate, then we cannot construct the covector $\omega = P^{-1}$ on M. In order to deal with this possibility, we change setting from the cotangent to the tangent bundle.

With g_0 and ∇^0 as above, suppose that M is equipped with a (possibly degenerate) Poisson tensor P, and let $K = g_0^{-1} + P$. If $\nabla^0 K = 0$, then $\nabla^0 P = 0$ and M is a regular Poisson manifold with symplectic foliation $\mathcal{S}(M)$ defined by the kernel of P (see [9]). The restriction P_S of P to a symplectic leaf S is nondegenerate, and S is endowed with a symplectic form $\omega = P_S^{-1}$.

A classical result of Lichnerowicz [5] states that there exist local coordinates x^i along $\mathcal{S}(M)$ and y^i along $\mathcal{N}$ (the transverse foliation orthogonal to $\mathcal{S}(M)$) in which

g_0 and ω have the form $g_0 = g' + g''$ where

$$g' = (g_0)_{ij}(y)dy^i dy^j, \quad g'' = (g_0)_{ij}(x)dx^i dx^j, \quad \omega = \omega_{ij}(x)dx^i \wedge dx^j.$$

By restricting ω and g'' to a symplectic leaf S, we are in the situation described by Shubin's Theorem above. Thus, we have a complex structure J which defines a Hermitian metric $g_s = g_{ij}(x)dx^i dx^j$ on S which is parallel with respect to g'' (and, therefore, with respect to g_0). We can extend J by 0 to all of M, and define a new metric $\tilde{g}$ on M by the formula $\tilde{g} = g' + g_s$.

This metric is called a *partially Kähler* metric. It is parallel with respect to g_0 and, when restricted to the symplectic leaf S, is a Hermitian metric on S. In his book [9], Vaisman concludes from these remarks that "the parallel Poisson structures of a Riemannian manifold (M, g_0) are exactly those defined by the Kähler foliation of the g_0-parallel partially-Kähler metrics of M (if any)". One can view this statement as the following generalization of Shubin's Theorem.

THEOREM. *If $\nabla^0 K = 0$, then M is a regular Poisson manifold (with the Poisson tensor P), and there exists a complex structure J on the symplectic leaves of M such that the metric $\tilde{g}$ defined above is parallel with respect to g_0 (thus $\nabla^{\tilde{g}} = \nabla^0$), and the restriction of $\tilde{g}$ to the symplectic leaf S defines a Kähler foliation on M.*

3. Related Questions

We have shown that the preservation of both Riemannian and Poisson structures on M by a linear connection imposes certain conditions on the connection itself, as well as on the structure of the manifold (M, g, P). In future work we will address some related questions, including: When can one guarantee the existence of a metriplectic connection on a manifold M, and is there an optimal or canonical choice of such a connection, similar to the canonical connection given in [3]?

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